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Analysis of stress and displacement fields in interlaminar-toughened composite laminates with transverse cracks

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Abstract—In recent years, to enhance the interlaminar fracture toughness of CFRP laminates, interlaminar-toughened laminates have been developed in which resin rich layers are placed in interlaminar regions. In the present study, stress and displacement fields in interlaminar-toughened cross-ply laminates with transverse cracks are analyzed. The analysis is based on a two-dimensional model which assumes that generalized plane strain conditions prevail. Account is taken of thermal residual stresses arising from a mismatch in thermal expansion coefficients of the 0° ply, interlaminar resin and 90° ply. The analytical results are shown for the laminates with interlaminar resin layers of various thicknesses. The analysis will be a basis for the optimal design of laminates with interlaminar resin layers.

Keywords: Composite material; crack opening displacement; interlaminar shear; CFRP; cross-ply laminate; transverse cracks; interlaminar-toughened laminate.

1. INTRODUCTION

In many applications of fiber-reinforced plastics, they are used mainly in the form of multidirectional laminates. Among these composites, cross-ply laminates have been extensively investigated [1–8] through both theoretical and experimental studies because the cross-ply laminate is a basic laminate configuration. The failure process of cross-ply laminates is known to involve an accumulation of transverse cracks and delamination.

Most previous studies on the failure process of cross-ply laminates have been focussed on the experimental measurement of transverse crack density as a function of applied load or on the theoretical prediction of the onset of transverse cracking

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and its multiplication. In conjunction with the theoretical prediction of mechanical behavior and damage progress of cracked laminates, micromechanical stress analyses of damaged cross-ply laminates have been conducted.

Takeda and Ogihara [1–4] conducted *in situ* observation of microscopic failure process in CFRP cross-ply laminates at room temperature and at 80°C. Progressive damage parameters, the transverse crack density and delamination ratio, were presented. They pointed out that the thermal residual stress has significant effect on the failure process and that quantitative estimation of thermal residual stress was important.

Hashin [5] conducted vatriational stress analysis for cracked cross-ply laminates under tension and shear, and predicted stiffness reduction. In the analysis, an admissible stress state was assumed and the principle of minimum complementary energy was applied. Nairn [6] extended Hashin's analysis to consider the thermal residual stresses and derived the energy release rate associated with transverse crack formation. The energy release rate was used to predict transverse crack density as a function of laminate stress.

Lee and Allen [7] proposed a mathematical model for predicting the stiffness reduction of cracked cross-ply laminates. In their analysis, the principle of minimum potential energy was utilized with assumed displacement fields. McCartney [8] analyzed stress transfer between 0° and 90° plies in cracked cross-ply laminates. The solutions were determined to satisfy the stress–strain–temperature relations either exactly or in an average sense.

In recent years, interlaminar-toughened laminates have been developed in which resin-rich layers are placed in interlaminar regions to enhance the interlaminar fracture toughness of CFRP laminates and to restrict delamination onset. Ozdil and Carlsson [9, 10] studied mode I interlaminar fracture toughness in laminates with interlaminar resin layers. They showed that the interlaminar layers considerably improved mode I interlaminar fracture toughness. Askoy and Carlsson [11] investigated mode II interlaminar fracture toughness in laminates with interlaminar layers.

Odagiri *et al.* [12, 13] developed a prepreg system which had fine polyamide particles on its surfaces. In laminates using the prepreg, polyamide particle-dispersed layers were formed at every ply interface. It was shown that the composite system had high interlaminar fracture toughness. It was also shown that interlaminar crack growth resistance under fatigue loading was higher [14, 15]. However, few studies of the failure process in these laminates considering transverse cracks have been conducted [16].

In the present study, stress and displacement fields in an interlaminar-toughened cross-ply laminate with transverse cracks are analyzed. The analysis is based on a two-dimensional model which assumes that generalized plane strain conditions prevail. Account is taken of thermal residual stresses arising from a mismatch in thermal expansion coefficients of the 0° ply, interlaminar resin and 90° ply. The analytical results are shown for the laminates with interlaminar resin layers of various thicknesses.

2. ANALYSIS

2.1. Analytical model

Consider an interlaminar-toughened symmetric cross-ply laminate as shown in Fig. 1 that has transverse cracks in 90° ply. A set rectangular Cartesian coordinates (x, y, z) is selected as shown in Fig. 1. The laminate is subjected to a tensile stress, σ, in the y-direction. It is assumed that a parallel array of equally spaced transverse cracks (crack spacing, $2L$) run through the thickness and width of the 90° ply. The transverse crack tips stop at the 90°/interlaminar resin layer interface. The thicknesses of the outer 0° ply, interlaminar resin layer and inner 90° ply are b, t and $2a$, respectively. The laminate thickness is denoted by $2h$, where $h = b + t + a$.

Following McCartney's analysis [8], generalized plane strain conditions are assumed such that the displacement components for the 0° ply, interlaminar resin layer and 90° ply are of the form

$$u = u(x, y), \qquad v = v(x, y), \qquad w = \varepsilon^* z, \tag{1}$$

where ε^* is the uniform transverse strain in the cracked laminate. The value of ε^* will be determined as part of the solution of the problem.

In the present study, two-dimensional general plane strain analysis is conducted for simplicity. However, in this condition, the laminate is subjected to a constraint in the transverse direction, that is, constraint strain of ε^*. This may not be a good approximation near the free edges. It is possible to analyze the three-dimensional problem if we add one more unknown function to $C(y)$ used in the present analysis.

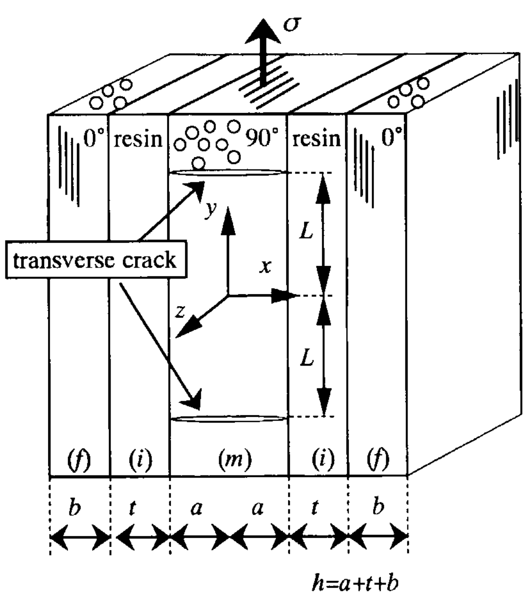

Figure 1. A model of interlaminar-toughened cross-ply laminate containing transverse cracks in 90° ply.

By taking a similar procedure to McCartney's [8], sets of the equilibrium equations and the stress–strain–temperature relations are obtained as following. Stress, strain and displacement components and thermo-elastic constants, which are associated with the 0° ply, are denoted by a superscript or subscript f. The corresponding quantities associated with the interlaminar resin layer and the 90° ply are denoted by a superscript or subscript, i and m, respectively. The symbols f and m were used by McCartney [8] and the symbol i is introduced in the present paper.

0° ply:

$$\frac{\partial\sigma^{\rm f}_{xx}}{\partial x}+\frac{\partial\sigma^{\rm f}_{xy}}{\partial y}=0,\qquad \frac{\partial\sigma^{\rm f}_{xy}}{\partial x}+\frac{\partial\sigma^{\rm f}_{yy}}{\partial y}=0, \tag{2}$$

$$\varepsilon^{\rm f}_{xx}=\frac{1}{E'^{\rm f}_{\rm T}}\sigma^{\rm f}_{xx}-\frac{\nu'^{\rm f}_{\rm A}}{E'^{\rm f}_{\rm A}}\sigma^{\rm f}_{yy}+\alpha'^{\rm f}_{\rm T}\Delta T-\nu^{\rm f}_{\rm T}\varepsilon^{*}\equiv\frac{\partial u_{\rm f}}{\partial x}, \tag{3}$$

$$\varepsilon^{\rm f}_{yy}=-\frac{\nu'^{\rm f}_{\rm A}}{E'^{\rm f}_{\rm A}}\sigma^{\rm f}_{xx}+\frac{1}{E'^{\rm f}_{\rm A}}\sigma^{\rm f}_{yy}+\alpha'^{\rm f}_{\rm A}\Delta T-\nu^{\rm f}_{\rm A}\frac{E^{\rm f}_{\rm T}}{E^{\rm f}_{\rm A}}\varepsilon^{*}\equiv\frac{\partial v_{\rm f}}{\partial y}, \tag{4}$$

$$\varepsilon^{\rm f}_{xy}=\frac{\sigma^{\rm f}_{xy}}{2\mu^{\rm f}_{\rm A}}\equiv\frac{1}{2}\left(\frac{\partial u_{\rm f}}{\partial y}+\frac{\partial v_{\rm f}}{\partial x}\right), \tag{5}$$

$$\begin{gathered}\frac{1}{E'^{\rm f}_{\rm T}}=\frac{1-\left(\nu^{\rm f}_{\rm T}\right)^{2}}{E^{\rm f}_{\rm T}},\qquad \frac{\nu'^{\rm f}_{\rm A}}{E'^{\rm f}_{\rm A}}=\frac{\nu^{\rm f}_{\rm A}\left(1+\nu^{\rm f}_{\rm T}\right)}{E^{\rm f}_{\rm A}},\\ \frac{1}{E'^{\rm f}_{\rm A}}=\frac{1}{E^{\rm f}_{\rm A}}\left\{1-\left(\nu^{\rm f}_{\rm A}\right)^{2}\frac{E^{\rm f}_{\rm T}}{E^{\rm f}_{\rm A}}\right\},\end{gathered} \tag{6}$$

$$\alpha'^{\rm f}_{\rm T}=\left(1+\nu^{\rm f}_{\rm T}\right)\alpha^{\rm f}_{\rm T},\qquad \alpha'^{\rm f}_{\rm A}=\alpha^{\rm f}_{\rm A}+\nu^{\rm f}_{\rm A}\frac{E^{\rm f}_{\rm T}}{E^{\rm f}_{\rm A}}\alpha^{\rm f}_{\rm T}, \tag{7}$$

interlaminar resin layer:

$$\frac{\partial\sigma^{\rm i}_{xx}}{\partial x}+\frac{\partial\sigma^{\rm i}_{xy}}{\partial y}=0,\qquad \frac{\partial\sigma^{\rm i}_{xy}}{\partial x}+\frac{\partial\sigma^{\rm i}_{yy}}{\partial y}=0, \tag{8}$$

$$\varepsilon^{\rm i}_{xx}=\frac{\sigma^{\rm i}_{xx}}{E'^{\rm i}}-\frac{\nu'^{\rm i}}{E'^{\rm i}}\sigma^{\rm i}_{yy}+\alpha'^{\rm i}\Delta T-\nu^{\rm i}\varepsilon^{*}\equiv\frac{\partial u_{\rm i}}{\partial x}, \tag{9}$$

$$\varepsilon^{\rm i}_{yy}=-\frac{\nu'^{\rm i}}{E'^{\rm i}}\sigma^{\rm i}_{xx}+\frac{\sigma^{\rm i}_{xx}}{E'^{\rm i}}+\alpha'^{\rm i}\Delta T-\nu^{\rm i}\varepsilon^{*}\equiv\frac{\partial v_{\rm i}}{\partial y}, \tag{10}$$

$$\varepsilon^{\rm i}_{xy}=\frac{\sigma^{\rm i}_{xy}}{2\mu^{\rm i}}\equiv\frac{1}{2}\left(\frac{\partial u_{\rm i}}{\partial y}+\frac{\partial v_{\rm i}}{\partial x}\right), \tag{11}$$

$$\frac{1}{E'^{\mathrm{i}}} = \frac{1 - \nu^{\mathrm{i}2}}{E'^{\mathrm{i}}}, \qquad \nu'^{\mathrm{i}} = \frac{\nu^{\mathrm{i}}}{1 - \nu^{\mathrm{i}}}, \tag{12}$$

$$\alpha'^{\mathrm{i}} = \alpha^{\mathrm{i}}(1 + \nu^{\mathrm{i}}), \tag{13}$$

90° ply:

$$\frac{\partial \sigma_{xx}^{\mathrm{m}}}{\partial x} + \frac{\partial \sigma_{xy}^{\mathrm{m}}}{\partial y} = 0, \qquad \frac{\partial \sigma_{xy}^{\mathrm{m}}}{\partial x} + \frac{\partial \sigma_{yy}^{\mathrm{m}}}{\partial y} = 0, \tag{14}$$

$$\varepsilon_{xx}^{\mathrm{m}} = \frac{1}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{xx}^{\mathrm{m}} - \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{yy}^{\mathrm{m}} + \alpha'^{\mathrm{m}}_{\mathrm{T}}\Delta T - \nu_{\mathrm{A}}^{\mathrm{m}}\varepsilon^* \equiv \frac{\partial u_{\mathrm{m}}}{\partial x}, \tag{15}$$

$$\varepsilon_{yy}^{\mathrm{m}} = -\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{xx}^{\mathrm{m}} + \frac{1}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{yy}^{\mathrm{m}} + \alpha'^{\mathrm{m}}_{\mathrm{T}}\Delta T - \nu_{\mathrm{A}}^{\mathrm{m}}\varepsilon^* \equiv \frac{\partial v_{\mathrm{m}}}{\partial y}, \tag{16}$$

$$\varepsilon_{xy}^{\mathrm{m}} = \frac{\sigma_{xy}^{\mathrm{m}}}{2\mu_{\mathrm{T}}^{\mathrm{m}}} \equiv \frac{1}{2}\left(\frac{\partial u_{\mathrm{m}}}{\partial y} + \frac{\partial v_{\mathrm{m}}}{\partial x}\right), \tag{17}$$

$$\frac{1}{E'^{\mathrm{m}}_{\mathrm{T}}} = \frac{1}{E_{\mathrm{T}}^{\mathrm{m}}}\left\{1 - \left(\nu_{\mathrm{A}}^{\mathrm{m}}\right)^2 \frac{E_{\mathrm{T}}^{\mathrm{m}}}{E_{\mathrm{A}}^{\mathrm{m}}}\right\}, \qquad \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}} = \frac{1}{E_{\mathrm{T}}^{\mathrm{m}}}\left\{\nu_{\mathrm{T}}^{\mathrm{m}} + \left(\nu_{\mathrm{A}}^{\mathrm{m}}\right)^2 \frac{E_{\mathrm{T}}^{\mathrm{m}}}{E_{\mathrm{A}}^{\mathrm{m}}}\right\}, \tag{18}$$

$$\alpha'^{\mathrm{m}}_{\mathrm{T}} = \alpha_{\mathrm{T}}^{\mathrm{m}} + \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{A}}^{\mathrm{m}}. \tag{19}$$

In the relations $\Delta T = T - T_0$, where T is the test temperature, and T_0 is the stress-free temperature. The parameters $E_{\mathrm{A}}, \nu_{\mathrm{A}}, \mu_{\mathrm{A}}$ and α_{A} are axial Young's modulus, Poisson's ratio, shear modulus and thermal expansion coefficient, and $E_{\mathrm{T}}, \nu_{\mathrm{T}}, \mu_{\mathrm{T}}$ and α_{T} are the corresponding transverse properties. It should be noted that $E_{\mathrm{T}} = 2\mu_{\mathrm{T}}(1 + \nu_{\mathrm{T}})$ but $E_{\mathrm{A}} \neq 2\mu_{\mathrm{A}}(1 + \nu_{\mathrm{A}})$.

2.2. *Boundary conditions*

The stress and displacement fields to be analyzed are only for the region between two neighboring transverse cracks. The origin for the rectangular Cartesian coordinates is selected to be midway between the two transverse cracks and the center of the 90° ply. Only solutions which are symmetrical about the x- and y-axes will be considered. It is therefore only necessary to consider the positive quadrant $x \geqslant 0$, $y \geqslant 0$.

Perfect bonding on the interfaces is assumed so that

$$\sigma_{xx}^{\mathrm{m}} = \sigma_{xx}^{\mathrm{i}}, \quad \sigma_{xy}^{\mathrm{m}} = \sigma_{xy}^{\mathrm{i}}, \quad \text{on } x = a, \ |y| \leqslant L, \tag{20}$$

$$u_{\mathrm{m}} = u_{\mathrm{i}}, \quad v_{\mathrm{m}} = v_{\mathrm{i}}, \quad \text{on } x = a, \ |y| \leqslant L, \tag{21}$$

$$\sigma_{xx}^{\mathrm{i}} = \sigma_{xx}^{\mathrm{f}}, \quad \sigma_{xy}^{\mathrm{i}} = \sigma_{xy}^{\mathrm{f}}, \quad \text{on } x = a + t, \ |y| \leqslant L, \tag{22}$$

$$u_{\mathrm{i}} = u_{\mathrm{f}}, \quad v_{\mathrm{i}} = v_{\mathrm{f}}, \quad \text{on } x = a + t, \ |y| \leqslant L. \tag{23}$$

The external surface of the 0° ply is stress-free so that

$$\sigma_{xx}^{\mathrm{f}} = 0, \quad \sigma_{xy}^{\mathrm{f}} = 0, \quad \text{on } x = h, \ |y| \leqslant L. \tag{24}$$

Because there is symmetry about the y-axis it follows that

$$\sigma_{xy}^{\mathrm{m}} = 0, \quad u_{\mathrm{m}} = 0, \quad \text{on } x = 0, \ |y| \leqslant L. \tag{25}$$

The symmetry about the x-axis asserts that

$$\sigma_{xy}^{\mathrm{m}} = 0, \quad \text{on } y = 0, \ 0 \leqslant x \leqslant a, \tag{26}$$

$$\sigma_{xy}^{\mathrm{i}} = 0, \quad \text{on } y = 0, \ a \leqslant x \leqslant a + t, \tag{27}$$

$$\sigma_{xy}^{\mathrm{f}} = 0, \quad \text{on } y = 0, \ a + t \leqslant x \leqslant h, \tag{28}$$

$$v_{\mathrm{m}} = 0, \quad \text{on } y = 0, \ 0 \leqslant x \leqslant a, \tag{29}$$

$$v_{\mathrm{i}} = 0, \quad \text{on } y = 0, \ a \leqslant x \leqslant a + t, \tag{30}$$

$$v_{\mathrm{f}} = 0, \quad \text{on } y = 0, \ a + t \leqslant x \leqslant h. \tag{31}$$

On the surface of the transverse crack

$$\sigma_{yy}^{\mathrm{m}} = 0, \quad \sigma_{xy}^{\mathrm{m}} = 0, \quad \text{on } y = L, \ 0 \leqslant x \leqslant a, \tag{32}$$

and for the interlaminar resin layer and outer 0° ply

$$\sigma_{xy}^{\mathrm{i}} = 0, \quad \text{on } y = L, \ a \leqslant x \leqslant a + t, \tag{33}$$

$$v_{\mathrm{i}} = L\varepsilon_{\mathrm{c}}, \quad \text{on } y = L, \ a \leqslant x \leqslant a + t, \tag{34}$$

$$\sigma_{xy}^{\mathrm{f}} = 0, \quad \text{on } y = L, \ a + t \leqslant x \leqslant h, \tag{35}$$

and

$$v_{\mathrm{f}} = L\varepsilon_{\mathrm{c}}, \quad \text{on } y = L, \ a \leqslant x \leqslant a + t, \tag{36}$$

where ε_{c} is the longitudinal strain applied to the cracked laminate.

If σ is the stress applied to the laminate in the y-direction, then the following condition is always satisfied

$$h\sigma = \int_0^a \sigma_{yy}^{\mathrm{m}} \, \mathrm{d}x + \int_a^{a+t} \sigma_{yy}^{\mathrm{i}} \, \mathrm{d}x + \int_{a+t}^h \sigma_{yy}^{\mathrm{f}} \, \mathrm{d}x. \tag{37}$$

The average transverse stress σ_t applied to the cracked laminate is assumed to be zero so that

$$hL\sigma_t = \int_0^a \int_0^L \sigma_{zz}^{\mathrm{m}} \, \mathrm{d}y\mathrm{d}x + \int_a^{a+t} \int_0^L \sigma_{zz}^{\mathrm{i}} \, \mathrm{d}y\mathrm{d}x + \int_{a+t}^h \int_0^L \sigma_{zz}^{\mathrm{f}} \, \mathrm{d}y\mathrm{d}x = 0. \tag{38}$$

This condition ensures that in the transverse direction the laminate does not support any load.

2.3. An approximate 2D solution

Following McCartney's analysis [8], it is assumed that the normal stress components in the y-direction are independent of x. In the present study, it is also assumed that the normal stress component in the y-direction in the interlaminar resin layer does not change after transverse cracking. The assumed stress components are

$$\sigma_{yy}^{\mathrm{f}} = C(y) + \sigma_{\mathrm{f}}, \quad \sigma_{\mathrm{f}} = E'^{\mathrm{f}}_{\mathrm{A}}\left(\varepsilon + \nu_{\mathrm{A}}^{\mathrm{f}}\frac{E_{\mathrm{T}}^{\mathrm{f}}}{E_{\mathrm{A}}^{\mathrm{f}}}\varepsilon^{*} - \alpha'^{\mathrm{f}}_{\mathrm{A}}\Delta T\right), \tag{39}$$

$$\sigma_{yy}^{\mathrm{i}} = \sigma_{\mathrm{i}}, \quad \sigma_{\mathrm{i}} = E'^{\mathrm{i}}\left(\varepsilon + \nu^{\mathrm{i}}\varepsilon^{*} - \alpha'^{\mathrm{i}}\Delta T\right), \tag{40}$$

$$\sigma_{yy}^{\mathrm{m}} = -\frac{b}{a}C(y) + \sigma_{\mathrm{m}}, \quad \sigma_{\mathrm{m}} = E'^{\mathrm{f}}_{\mathrm{T}}\left(\varepsilon + \nu_{\mathrm{A}}^{\mathrm{m}}\varepsilon^{*} - \alpha'^{\mathrm{f}}_{\mathrm{T}}\Delta T\right), \tag{41}$$

where $C(y)$ is an unknown function to be determined and where strains ε and ε^{*} are to be chosen so that (37) and (38) are satisfied. The corresponding results are shown in Appendix. σ_{f}, σ_{i} and σ_{m} are the stresses in the 0° ply, interlaminar resin layer and 90° ply, respectively, that would result in a damage-free laminate having a longitudinal strain ε and a transverse strain ε^{*}.

On substituting (39), (40) and (41) into $(2)_2$, $(8)_2$ and $(14)_2$, respectively, and integrating with respect to x subject to the boundary conditions $(20)_2$, $(22)_2$, $(24)_2$ and $(25)_1$, it is easily shown that

$$\sigma_{xy}^{\mathrm{f}} = C'(y)(h - x), \tag{42}$$

$$\sigma_{xy}^{\mathrm{m}} = \frac{b}{a}C'(y)x, \tag{43}$$

$$\sigma_{xy}^{\mathrm{i}} = bC'(y). \tag{44}$$

On substitution of (42), (43) and (44) into $(2)_1$, $(8)_2$ and $(14)_1$, respectively, and integrating with respect to x subject to the boundary conditions $(24)_1$, $(20)_1$ and $(22)_1$, it is shown that

$$\sigma_{xx}^{\mathrm{f}} = \frac{1}{2}C''(y)(h - x)^2, \tag{45}$$

$$\sigma_{xx}^{\mathrm{i}} = bC''(y)\left(-x + a + t + \frac{b}{2}\right), \tag{46}$$

$$\sigma_{xx}^{\mathrm{m}} = \frac{b}{2a}C''(y)\left\{a(h + t) - x^2\right\}. \tag{47}$$

The substitution of (41) and (47) into (15) and integrating with respect to x subject to the boundary condition $(25)_2$ leads to the result

$$u_{\mathrm{m}} = \frac{b}{a}\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}C(y)x - \frac{1}{6}\frac{b}{a}\frac{1}{E'^{\mathrm{m}}_{\mathrm{T}}}C''(y)\{x^3 - 3a(h+t)x\}$$
$$- \left(\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{\mathrm{m}} - \alpha'^{\mathrm{m}}_{\mathrm{T}}\Delta T + \nu^{\mathrm{m}}_{\mathrm{A}}\varepsilon^*\right)x. \tag{48}$$

On substituting (40) and (46) into (9) and integrating with respect to x subject to the boundary condition $(21)_1$ it can be shown that

$$u_{\mathrm{i}} = C''(y)\left[\frac{b}{E'^{\mathrm{i}}}\left\{-\frac{1}{2}(x^2 - a^2) + \left(a + t + \frac{b}{2}\right)(x-a)\right\}\right.$$
$$\left. - \frac{b}{6}\frac{a}{E'^{\mathrm{m}}_{\mathrm{T}}}\{a - 3(h+t)\}\right]$$
$$- \left(\frac{\nu'^{\mathrm{i}}}{E'^{\mathrm{i}}}\sigma_{\mathrm{i}} - \alpha'^{\mathrm{i}}\Delta T + \nu^{\mathrm{i}}\varepsilon^*\right)(x-a) + \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}bC(y)$$
$$- \left(\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{\mathrm{m}} - \alpha'^{\mathrm{m}}_{\mathrm{T}}\Delta T + \nu^{\mathrm{m}}_{\mathrm{A}}\varepsilon^*\right)a. \tag{49}$$

On substituting (29) and (45) into (3) and integrating with respect to x subject to the boundary condition $(23)_1$ it can be shown that

$$u_{\mathrm{f}} = C(y)\left[\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}b - \frac{\nu'^{\mathrm{f}}_{\mathrm{A}}}{E'^{\mathrm{f}}_{\mathrm{A}}}\{x - (a+t)\}\right]$$
$$+ C''(y)\left[\frac{1}{6E'^{\mathrm{f}}_{\mathrm{T}}}\{b^3 - (h-x)^3\} + \frac{bt(t+b)}{2E'^{\mathrm{i}}} - \frac{b}{6}\frac{a}{E'^{\mathrm{m}}_{\mathrm{T}}}\{a - 3(h+t)\}\right]$$
$$- \left(\frac{\nu'^{\mathrm{f}}_{\mathrm{A}}}{E'^{\mathrm{f}}_{\mathrm{A}}}\sigma_{\mathrm{f}} - \alpha'^{\mathrm{f}}_{\mathrm{T}}\Delta T + \nu^{\mathrm{f}}_{\mathrm{T}}\varepsilon^*\right)\{x - (a+t)\}$$
$$- \left(\frac{\nu'^{\mathrm{i}}}{E'^{\mathrm{i}}}\sigma_{\mathrm{i}} - \alpha'^{\mathrm{i}}\Delta T + \nu^{\mathrm{i}}\varepsilon^*\right)t$$
$$- \left(\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\sigma_{\mathrm{m}} - \alpha'^{\mathrm{m}}_{\mathrm{T}}\Delta T + \nu^{\mathrm{m}}_{\mathrm{A}}\varepsilon^*\right)a. \tag{50}$$

On differentiating (50) with respect to y and substituting into (5), an expression for $\partial v_{\mathrm{f}}/\partial x$ can be derived. On integrating the resulting expression with respect to x it can be shown that

$$\begin{aligned} v_{\mathrm{f}} = C'(y)\Bigg[&\frac{1}{\mu_{\mathrm{A}}^{\mathrm{f}}}\bigg\{h\{x-(a+t)\} - \frac{1}{2}\{x^2-(a+t)^2\}\bigg\} \\ &- \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}b\{x-(a+t)\} \\ &+ \frac{\nu'^{\mathrm{f}}_{\mathrm{A}}}{E'^{\mathrm{f}}_{\mathrm{A}}}\bigg\{\frac{1}{2}\{x^2-(a+t)^2\} - (a+t)\{x-(a+t)\}\bigg\}\Bigg] \\ - C'''(y)\Bigg[&\frac{1}{6E'^{\mathrm{f}}_{\mathrm{T}}}\bigg\{b^3\{x-(a+t)\} + \frac{1}{4}\{(h-x)^4 - b^4\}\bigg\} \\ &+ \frac{bt(t+b)}{2E'^{\mathrm{i}}}\{x-(a+t)\} \\ &- \frac{ab}{6E'^{\mathrm{m}}_{\mathrm{T}}}\{a-3(h+t)\}\{x-(a+t)\}\Bigg] + A_1(y), \end{aligned} \tag{51}$$

where $A_1(y)[= v_{\mathrm{f}}(a+t, y)]$ is, for a moment, an arbitrary function arising from the integration. On differentiating (49) with respect to y and substituting into (11), an expression for $\partial v_{\mathrm{i}}/\partial x$ can be derived. On integrating the resulting expression with respect to x it can be shown that

$$\begin{aligned} v_{\mathrm{i}} = \left(\frac{1}{\mu^{\mathrm{i}}} - \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\right)&b(x-a)C'(y) \\ - C'''(y)\Bigg[&\frac{b}{E'^{\mathrm{i}}}\bigg\{-\frac{1}{6}\left(x^3-a^3\right) + \frac{1}{2}a^2(x-a) \\ &+ \left(a+t+\frac{b}{2}\right)\bigg\{\frac{1}{2}\left(x^2-a^2\right) - a(x-a)\bigg\}\bigg\} \\ &- \frac{ab}{6E'^{\mathrm{m}}_{\mathrm{T}}}\{a-3(h+t)\}(x-a)\Bigg] + A_2(y), \end{aligned} \tag{52}$$

where $A_2(y)[= v_{\mathrm{i}}(a, y)]$ is, for a moment, an arbitrary function arising from the integration. On differentiating (48) with respect to y and substituting into (17), an expression for $\partial v_{\mathrm{m}}/\partial x$ can be derived. On integrating the resulting expression with respect to x it can be shown that

$$v_{\mathrm{m}} = \frac{b}{2a}\left(\frac{1}{\mu_{\mathrm{T}}^{\mathrm{m}}} - \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\right)(x^2 - a^2)C'(y) + \frac{b}{6aE'^{\mathrm{m}}_{\mathrm{T}}}C'''(y)\left\{\frac{1}{4}(x^4 - a^4) - \frac{3}{2}a(h+t)(x^2 - a^2)\right\} + A_2(y). \quad (53)$$

From (52) and (53), boundary condition $(21)_2$ is satisfied. Using boundary condition $(23)_2$, the relation between $A_1(y)$ and $A_2(y)$ is derived as

$$A_1(y) = \left(\frac{1}{\mu^{\mathrm{i}}} - \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\right)btC'(y) - \left[\frac{b}{E'^{\mathrm{i}}}\left(\frac{t^3}{3} + \frac{bt^2}{4}\right) - \frac{abt}{6E'^{\mathrm{m}}_{\mathrm{T}}}\{a - 3(h+t)\}\right]C'''(y) + A_2(y). \quad (54)$$

Although the solution specified by relations (39)–(53) satisfies the stress–strain–temperature relations (2), (3), (5), (8), (9), (11), (14), (15) and (17) it cannot exactly satisfy relations (4), (10) and (16), which have not, so far, been used in the analysis. Following the approach of McCartney [8], relations (4), (10) and (16) are replaced by their averages. If $f(x, y), i(x, y)$ and $m(x, y)$ are any variables associated with 0° ply, interlaminar resin layer and 90° ply, averages can be defined by

$$\overline{f}(y) = \frac{1}{b}\int_{a+t}^{h} f(x, y)\,\mathrm{d}x, \quad (55)$$

$$\overline{i}(y) = \frac{1}{t}\int_{a}^{a+t} i(x, y)\,\mathrm{d}x, \quad (56)$$

$$\overline{m}(y) = \frac{1}{a}\int_{0}^{a} m(x, y)\,\mathrm{d}x. \quad (57)$$

Thus on averaging, relations (4), (10) and (16) may be written in the following respective forms:

$$\frac{\mathrm{d}\overline{v}_{\mathrm{f}}(y)}{\mathrm{d}y} = -\frac{\nu'^{\mathrm{f}}_{\mathrm{A}}}{E'^{\mathrm{f}}_{\mathrm{A}}}\overline{\sigma}^{\mathrm{f}}_{xx} + \frac{1}{E'^{\mathrm{f}}_{\mathrm{A}}}\overline{\sigma}^{\mathrm{f}}_{yy} + \alpha'^{\mathrm{f}}_{\mathrm{A}}\Delta T - \nu^{\mathrm{f}}_{\mathrm{A}}\frac{E^{\mathrm{f}}_{\mathrm{T}}}{E^{\mathrm{f}}_{\mathrm{A}}}\varepsilon^*, \quad (58)$$

$$\frac{d\overline{v}_i(y)}{dy} = -\frac{\nu'^i}{E'^i}\overline{\sigma}^i_{xx} + \frac{\overline{\sigma}^i_{yy}}{E'^i} + \alpha'^i\Delta T - \nu^i\varepsilon^*, \tag{59}$$

$$\frac{d\overline{v}_m(y)}{dy} = -\frac{\nu'^m_T}{E'^m_T}\overline{\sigma}^m_{xx} + \frac{1}{E'^m_T}\overline{\sigma}^m_{yy} + \alpha'^m_T\Delta T - \nu^m_A\varepsilon^*. \tag{60}$$

Furthermore, for the interlaminar resin layer and 0° ply, an averaging function

$$\overline{i + f(y)} = \frac{1}{t+b}\{t\overline{i}(y) + b\overline{f}(y)\}, \tag{61}$$

is defined and (58)–(59) are combined into one equation as

$$\left\{t\frac{d\overline{v}_i(y)}{dy} + b\frac{d\overline{v}_f(y)}{dy}\right\} = b\left(-\frac{\nu'^f_A}{E'^f_A}\overline{\sigma}^f_{xx} + \frac{1}{E'^f_A}\overline{\sigma}^f_{yy} + \alpha'^f_A\Delta T - \nu^f_A\frac{E^f_T}{E^f_A}\varepsilon^*\right)$$
$$+ t\left(-\frac{\nu'^i}{E'^i}\overline{\sigma}^i_{xx} + \frac{\overline{\sigma}^i_{yy}}{E'^i} + \alpha'^i\Delta T - \nu^i\varepsilon^*\right). \tag{62}$$

On substituting (39)–(41), (45)–(47) and (51)–(53) into (60) and (62), followed by subtraction to eliminate the function $A_2(y)$, the following ordinary differential equation is obtained:

$$Fb^4C''''(y) - Gb^2C''(y) + HC(y) = 0, \tag{63}$$

where

$$F = \frac{1}{b^4}\left(\frac{b}{t+b}\left[\frac{b^4}{20E^f_T} + \frac{b^2t(t+b)}{4E'^i} - \frac{ab}{6E'^m_T}\{a - 3(h+t)\}\left(\frac{b}{2} + t\right)\right.\right.$$
$$\left. + \frac{bt^2}{E'^i}\left(\frac{t}{3} + \frac{b}{4}\right)\right]$$
$$+ \frac{t}{t+b}\left[\frac{bt^2(3t+2b)}{24E'^i} - \frac{abt}{12E'^m_T}\{a - 3(h+t)\}\right]$$
$$\left. + \frac{a^2b}{6E'^m_T}\left(-\frac{a}{5} + h + t\right)\right), \tag{64}$$

$$G = \frac{1}{b^2}\left[\frac{b^2}{t+b}\left\{\frac{b}{3\mu^f_A} + \frac{b\nu'^f_A}{3E'^f_A} + \left(\frac{1}{\mu^i} - \frac{\nu'^m_T}{E'^m_T}\right)t - \frac{b\nu'^m_T}{2E'^m_T}\right\}\right.$$

$$+\frac{t}{t+b}\left\{\left(\frac{1}{\mu^{\mathrm{i}}}-\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\right)\frac{bt}{2}+\frac{\nu'^{\mathrm{i}}b(t+b)}{2E'^{\mathrm{i}}}\right\}$$

$$+\frac{ab}{3}\left(\frac{1}{\mu^{\mathrm{m}}_{\mathrm{T}}}-\frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{E'^{\mathrm{m}}_{\mathrm{T}}}\right)-\frac{b\nu'^{\mathrm{m}}_{\mathrm{T}}}{2E'^{\mathrm{m}}_{\mathrm{T}}}\left(h+t-\frac{a}{3}\right)\Bigg], \tag{65}$$

$$H=b\left(\frac{1}{(t+b)E'^{\mathrm{f}}_{\mathrm{A}}}+\frac{1}{aE'^{\mathrm{m}}_{\mathrm{T}}}\right). \tag{66}$$

The general solution of the differential equation (63) satisfying the symmetry condition $C(y)=C(-y)$ is given by
if $s>r$:

$$C(y)=P\cosh\frac{py}{b}\cos\frac{qy}{b}+Q\sinh\frac{py}{b}\sin\frac{qy}{b}, \tag{67}$$

if $s=r$:

$$C(y)=P\cosh\frac{py}{b}+Q\frac{y}{b}\sinh\frac{py}{b}, \tag{68}$$

if $s<r$:

$$C(y)=P\cosh\frac{py}{b}\cosh\frac{qy}{b}+Q\sinh\frac{py}{b}\sinh\frac{qy}{b}, \tag{69}$$

where

$$p=\left[\frac{1}{2}(r+s)\right]^{\frac{1}{2}},\quad q=\left[\frac{1}{2}|r-s|\right]^{\frac{1}{2}}, \tag{70}$$

and

$$r=\frac{G}{2F},\quad s=\left(\frac{H}{F}\right)^{\frac{1}{2}}. \tag{71}$$

It follows from (67)–(69) that $C'(0)=0$ so that, on using (42)–(44), the boundary conditions (26)–(28) are automatically satisfied. It follows from (42)–(44) that conditions $(32)_2$, (33) and (35) are satisfied if

$$C'(L)=0,\quad C(L)=\frac{a}{b}\sigma_{\mathrm{m}}. \tag{72}$$

Conditions (72) are sufficient to determine the values of the parameters P and Q appearing in the solution (67)–(69). It can be shown that for $s>r$ (other cases are easily derived)

$$P=\Lambda\left(q\sinh\frac{pL}{b}\cosh\frac{qL}{b}+p\cosh\frac{pL}{b}\sinh\frac{qL}{b}\right)\frac{a}{b}\sigma_{\mathrm{m}}, \tag{73}$$

$$Q=-\Lambda\left(p\sinh\frac{pL}{b}\cosh\frac{qL}{b}+q\cosh\frac{pL}{b}\sinh\frac{qL}{b}\right)\frac{a}{b}\sigma_{\mathrm{m}}, \tag{74}$$

where

$$\frac{1}{\Lambda} = p \sinh \frac{qL}{b} \cosh \frac{qL}{b} + q \cosh \frac{pL}{b} \sinh \frac{pL}{b}. \tag{75}$$

The only boundary conditions which have not been satisfied are given by (29)–(31), (34) and (36), although the last is strictly a definition for the strain applied to the cracked laminate. It is clear from (51)–(53) that it is not possible for the approximate solution derived in this paper to satisfy the conditions (29)–(31) exactly. Following McCartney [8], conditions (29)–(31) are replaced by the following averaged conditions

$$\overline{v}_{\mathrm{m}}(0) = \overline{v}_{\mathrm{i}}(0) = \overline{v}_{\mathrm{f}}(0) = 0, \tag{76}$$

$$\overline{v}_{\mathrm{i}}(L) = \overline{v}_{\mathrm{f}}(L) = L\varepsilon_{\mathrm{c}}. \tag{77}$$

From (72), the conditions (76) are satisfied. Using (77), the function $A_2(y)$ is given by

$$\begin{aligned} A_2(y) = &-\frac{1}{E'^{\mathrm{m}}_{\mathrm{T}}}\frac{b}{a}\overline{C}(y) + \frac{b}{3}\left\{\frac{a}{\mu^{\mathrm{m}}_{\mathrm{T}}} - \frac{\nu'^{\mathrm{m}}_{\mathrm{T}}}{2E'^{\mathrm{m}}_{\mathrm{T}}}(a + 3h + 3t)\right\}C'(y) \\ &- \frac{a^2 b}{6E'^{\mathrm{m}}_{\mathrm{T}}}\left(-\frac{1}{5}a + h + t\right)C'''(y) + \varepsilon y. \end{aligned} \tag{78}$$

Thus, the stress and displacement fields in interlaminar-toughened cross-ply laminates with transverse cracks are derived.

3. RESULTS AND DISCUSSION

The material system analyzed is T800H/3631 carbon/epoxy system as CFRP prepreg and FM300 epoxy resin as interlaminar resin material. The thermo-elastic constants used in the analysis are listed in Table 1. The values for T800H/3631 are taken from [1] and those for FM300 are taken from [16]. In the present study, effects of interlaminar resin layer thickness are investigated.

The laminate configuration chosen is $(0/\mathrm{A}/90)_{\mathrm{s}}$, where A denotes the interlaminar resin layer. The ply thickness is 0.135 mm ($a = b = 0.135$ mm) and t is set to be $0.1a$, $0.5a$ and a. Transverse crack spacing, $2L = 10a$, and $\Delta T = -160$°C are used.

Figure 2 shows normal stress in the y-direction in the 90° ply as a function of y (distance from the midline between transverse cracks). The stress is normalized by σ^{m} which is the 90° ply stress in the damage-free laminate. At transverse crack surface ($y = L$), the stress is zero. The stress has a maximum value at $y = 0$. As the

Table 1.
Mechanical properties used in the analysis

Composite T800H/3631		Resin FM300	
E_A (GPa)	152.2	E^i (GPa)	2.45
E_T (GPa)	9.57	μ^i (GPa)	0.88
μ_A (GPa)	4.50	ν^i	0.38
μ_T (GPa)	3.21	$\alpha^i(\times 10^{-6})$	60.0
ν_A	0.349		
ν_T	0.490		
$\alpha_A(\times 10^{-6})$	0.100		
$\alpha_T(\times 10^{-6})$	35.5		

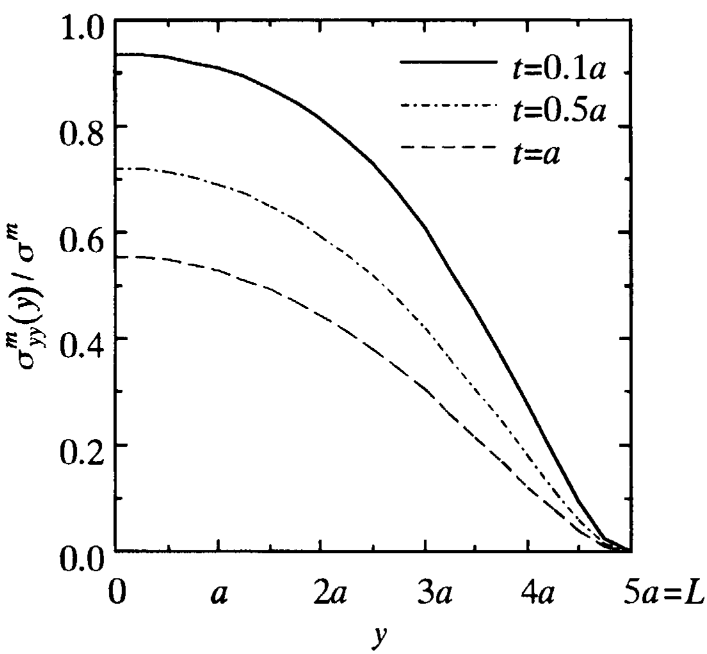

Figure 2. Distribution of normal stress in the y-direction in 90° ply.

interlaminar resin layer thickness increases, the maximum normalized normal stress in the 90° ply decreases.

Figure 3 shows shear stress in the interlaminar resin layer as a function of y. The value of shear stress divided by σ^m is shown. From the boundary conditions, the shear stress is zero at $y = 0$ and $y = L$. As the interlaminar resin layer thickness increases, the shear stress decreases, which corresponds to the dependence of the normal stress in the y-direction in 90° ply on the interlaminar resin layer thickness as shown in Fig. 2.

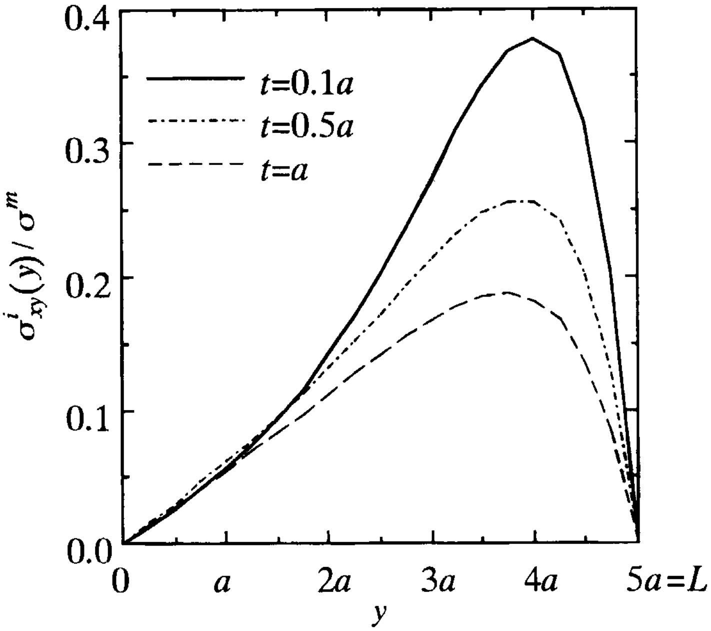

Figure 3. Distribution of shear stress in interlaminar resin layer.

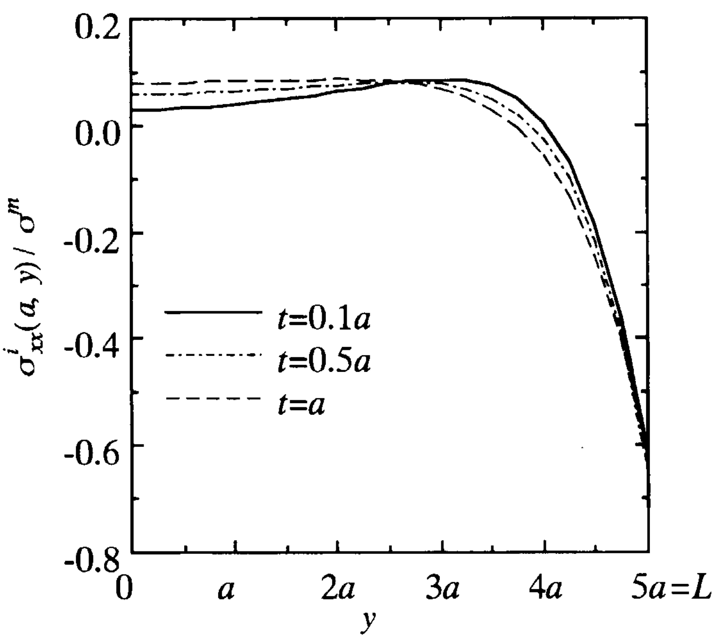

Figure 4. Distribution of normal stress in the x-direction along interlaminar resin layer/90° ply interface.

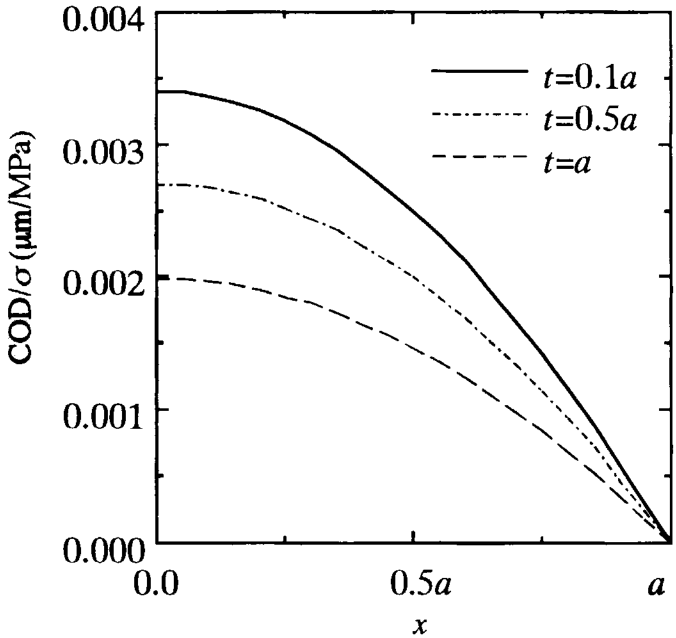

Figure 5. Transverse crack opening displacement as function of x.

Figure 4 shows normal stress in the x-(thickness) direction at the 90°/interlaminar resin layer interface. The stress is normalized with σ^{m}. The stress is tensile in the central region between the transverse cracks and compressive near the transverse cracks. The dependence of the stress distribution on the interlaminar resin layer thickness is small.

Figure 5 shows COD (crack opening displacement) of a transverse crack. The COD is obtained by using the following equation.

$$\mathrm{COD} = 2\{v_{\mathrm{m}}(a, L) - v_{\mathrm{m}}(x, L)\}. \tag{79}$$

The value of COD divided by σ is shown as a function of x. The position $x = 0$ corresponds to the center of the transverse crack and $x = a$ to the transverse crack tip. As the interlaminar resin layer thickness increases, COD/σ decreases.

4. CONCLUSION

The stress and displacement fields in an interlaminar-toughened cross-ply laminates with transverse cracks are analyzed. The effects of interlaminar resin layer thickness are investigated. Based on the analysis, it will be possible to predict the stiffness reduction due to transverse cracking and transverse crack formation in the laminates. The analysis will be a basis for the optimal design of laminates with interlaminar resin layers.

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APPENDIX

ε and ε^* in Section 2.1 are derived using (37) and (38). Following a similar procedure to McCartney's [8], the resulting expression is (for $s < r$)

$$\varepsilon = \frac{\sigma}{E_{\mathrm{A}}} - \frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\left(\nu_{\mathrm{A}}^{\mathrm{m}} - \nu_{\mathrm{A}}^{\mathrm{f}}\frac{E_{\mathrm{T}}^{\mathrm{f}}}{E_{\mathrm{A}}^{\mathrm{f}}}\right)\frac{ab}{hL}\Phi\sigma_{\mathrm{m}} + \alpha_{\mathrm{A}}\Delta T, \tag{A1}$$

$$\varepsilon^* = -\frac{\nu_{\mathrm{A}}}{E_{\mathrm{A}}}\sigma + \frac{1}{E_{\mathrm{T}}}\left(\nu_{\mathrm{A}}^{\mathrm{m}} - \nu_{\mathrm{A}}^{\mathrm{f}}\frac{E_{\mathrm{T}}^{\mathrm{f}}}{E_{\mathrm{A}}^{\mathrm{f}}}\right)\frac{ab}{hL}\Phi\sigma_{\mathrm{m}} + \alpha_{\mathrm{T}}\Delta T, \tag{A2}$$

$$\sigma_{\mathrm{m}} = \frac{E_{\mathrm{T}}'^{\mathrm{m}}}{\xi}\left\{\left(1 - \nu_{\mathrm{A}}\nu_{\mathrm{A}}^{\mathrm{m}}\right)\frac{\sigma}{E_{\mathrm{A}}} + \left(\alpha_{\mathrm{A}} + \nu_{\mathrm{A}}^{\mathrm{m}}\alpha_{\mathrm{T}} - \alpha_{\mathrm{T}}'^{\mathrm{m}}\right)\Delta T\right\}, \tag{A3}$$